On a blowup formula for sheaf-theoretic virtual enumerative invariants on projective surfaces

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Plan of talk:

1. Donaldson invariatns on closed four-manifolds
2. Mochizuki's virtual Donaldson invariants on projective surfaces
3. A blowup formula for sheaf-theoretic virtual enumerative invariants on projective surfaces

The anti-self-dual instanton moduli spaces

For now, let X be a closed, oriented, simply-connected, smooth four-manifold, and let $P \to X$ be a principal $SO(3)$-bundle with the first Pontryagin class p_1 and the second Stiefel–Whitney class w_2.

Fix a Riemannian metric g on X, and consider the Hodge star operator $*_g$ on $\Lambda^2_X := (\Lambda^2 T^* X)$. This satisfies $*_g^2 = 1$, so Λ^2_X decomposes as $\Lambda^2_X = \Lambda^+_X \oplus \Lambda^-_X$.

Definition: A connection A on P is said to be an *anti-self-dual instanton*, if the curvature F_A of A satisfies $F_A^+ := \pi_+(F_A) = 0$, where $\pi_+ : \Gamma(\mathfrak{g}_P \otimes \Lambda^2_X) \to \Gamma(\mathfrak{g}_P \otimes \Lambda^+_X)$ is the projection and $\mathfrak{g}_P$ is the adjoint bundle of P.

We denote by $\mathcal{A}_P$ the set of all connections on P and by $\mathcal{G}_P := \mathrm{Aut}(P)$ the set of all gauge transformations on P, or the *gauge group*.

The gauge group $\mathcal{G}_P$ acts on $\mathcal{A}_P$, and we consider the *anti-self-dual instanton moduli space*:

$$M_{X,g}(w_2, p_1) := \{A \in \mathcal{A}_P \, : \, F_A^+ = 0\}/\mathcal{G}_P.$$

This is an **oriented smooth manifold of expected dimensions** for a generic choice of Riemannian metrics of X, if $b_X^+ > 0$, where b_X^+ is the number of positive eigenvalues of the intersection form on $H^2(X, \mathbb{Z})$.

Uhlenbeck compactification:

$$\overline{M}_{X,g}(w_2, p_1) := \coprod_{\ell=0}^{-p_1/4} M_{X,g}(w_2, p_1 + 4\ell) \times S^\ell X,$$

where $S^\ell X$ is the ℓ-th symmetric product of X. This is equipped with a natural topology, and it is a compact Hausdorff space.

Donaldson's polynomial invariants

Assume $b_X^+ > 0$. Denote by 2d the expected dimension of $M_{X,g}(w_2, p_1)$ and by A_d the symmetric algebra of degree d generated by $H_2(X)$ and $H_0(X)$ with $\alpha \in H_2(X)$ degree 1 and $p \in H_0(X)$ degree 2.

One can define $\bar{\mu} : H_p(X, \mathbb{Z}) \to H^{4-p}(\overline{M}_{X,g}(w_2, p_1), \mathbb{Z})$ for $p = 0, 2$ via a universal bundle on the moduli space. We then define Donaldson's polynomial invariant $q_{X,d} : A_d \to \mathbb{Z}$ of degree d by

$$q_{X,d}(\sigma_1, \ldots, \sigma_n, p^m) := \int_{[\overline{M}_{X,g}(w_2,p_1)]} \bar{\mu}(\sigma_1) \cup \cdots \cup \bar{\mu}(\sigma_n) \cup \bar{\mu}(p)^m,$$

where $\sigma_1, \ldots, \sigma_n \in H_2(X)$, $p \in H_0(X)$ and $d = n + 2m$. This is independent of the choice of Riemannian metrics of X, if $b_X^+ > 1$.

If $w_2 \neq 0$, then there is no trivial bundle in lower strata of the Uhlenbeck compactification, and the fundamental class $[\overline{M}_{X,g}(w_2, p_1)]$ is well-defined. However, if $w_2 = 0$, then there is a trivial bundle in a lower stratum of the compactification. In this case, we introduce the notion of *stable range* (e.g. one requires $-p_1$ is sufficiently large) to have a well-defined fundamental class $[\overline{M}_{X,g}(w_2, p_1)]$.

Blowup formula by Friedman–Morgan: Consider the blowup $\widehat{X} \to X$ at a point in X and denote by e its exceptional divisor. Then Friedman and Morgan prove:

$$q_{\widehat{X},d+1}(\sigma_1, \ldots, \sigma_d, e, e, e, e) = -2q_{X,d}(\sigma_1, \ldots, \sigma_d).$$

This can be used to define the Donaldson invariants for *unstable range*.

Moreover, blowup formulae played fantastic roles in figuring out algebraic structures in the theory of the Donaldson invariants such as:

Kronheimer–Mrowka ('95) proved the structure theorem for the Donaldson invariants. Denote by $q_X : \oplus_d A_d \to \mathbb{Q}$ the polynomial invariants. They considered the following Donaldson series:

$$\sum \frac{q_X(\Sigma^d)}{d!} + \frac{1}{2}\sum \frac{q_X(p\Sigma^d)}{d!}.$$

under the assumption of X *simple type*, and proved that this formal series can be written by a finite collection of *basic classes* by developing a theory of singular connections and a blowup formula.

Fintsushel–Stern ('96) gave an alternative proof of the above structure theorem by improving the blowup formula. They further obtained a universal form of blowup formula, surprisingly, it is expressed in terms of modular forms.

Göttsche ('96) determined the wall-crossing term under the assumption that *Kotschick–Morgan conjecture* is true by an effective use of the blowup formula, and it turned out to be written in terms of modular forms.

Mochizuki's virtual Donaldson invariants

The **Hitchin–Kobayashi correspondence** tells the instanton moduli space is the same as the moduli space of semistable sheaves on a complex projective surface. (cf. Algebraic Donaldson invariants.) This is good, as we have a nice compactification of the latter moduli space thanks to **Gieseker** and **Maruyama**.

Then there is one other way of defining Donaldson type invariant from the moduli space of semistable sheaves in Algebraic Geometry by using *virtual techniques* developed by **Li–Tian** and **Behrend–Fantechi**. (cf. **Li–Tian** and **Fukaya–Ono** in Symplectic Geometry).

Perfect obstruction theory. Denote by $\mathcal{M}$ the moduli space of semistable sheaves on a projective surface X. We consider the case that semistability implies stability. Locally, $\mathcal{M}$ is the zero set of the Kuranishi map $\kappa : \mathrm{Ext}^1_0(E, E) \to \mathrm{Ext}^2_0(E, E)$ around $[E] \in \mathcal{M}$.

A moral here is to consider the following local model attached to it:

$$\begin{array}{ccc} & & \mathcal{V} \\ & & \big\downarrow \big\uparrow s \\ M := Z(s) & \subset & \mathcal{X}, \end{array}$$

where $\mathcal{X}$ is an ambient space, $\mathcal{V}$ is a bundle on it, and the moduli space is locally the zero set $Z(s)$ of a section s of $\mathcal{V}$.

This local model in fact yields a two-term locally-free resolution of the Kuranishi map, and by dualising the above geometric description one may package it into the following diagram:

$$\begin{array}{ccc} \mathcal{V}^\vee|_M & \xrightarrow{ds^\vee|_M} & \Omega_{\mathcal{X}}|_M \\ \big\downarrow s^\vee|_M & & \big\downarrow id \\ \mathcal{I}_{M/\mathcal{X}}/\mathcal{I}^2_{M/\mathcal{X}} & \longrightarrow & \Omega_{\mathcal{X}}|_M. \end{array}$$

Then Behrend and Fantechi consider a globalisation of this.

They define a *perfect obstruction theory* on a Deligne–Mumford stack $\mathbb{M}$ as a complex $E^\bullet \in D^b(\mathbb{M})$, locally quasi-isomorphic to a complex of locally-free sheaves supported in degree $[-1, 0]$ with a morphism $\phi : E^\bullet \to \mathbb{L}_{\mathbb{M}}^{\geq 1}$, where $\mathbb{L}_{\mathbb{M}}^{\geq 1}$ is $[-1, 0]$-truncation of the cotangent complex, so that $h^0(\phi)$ is isomorphism and $h^{-1}(\phi)$ is surjective. From these data, they construct a *virtual fundamental class* in the Chow group of $\mathbb{M}$ of the right degree and prove that it is a deformation invariant.

Then, one can use this class to define *virtual Donaldson invariants* on a projective surface X.

$$D_{X,c_1}(\sigma_1 \dots \sigma_n) := \int_{[\mathcal{M}]^{\mathrm{vir}}} \mu(\sigma_1) \cdot \dots \cdot \mu(\sigma_n),$$

where $\sigma_1, \dots, \sigma_n \in H^*(X)$, and $\mu : H^i(X) \to H^i(\mathcal{M})$, when the moduli space $\mathcal{M}$ does not contain strictly semistable sheaves.

Takuro Mochizuki performed this in full generality (even with parabolic structures). More precisely, Mochizuki

- formulates the invariants where the moduli space may have strictly semistable sheaves;
- proves a weak wall-crossing formula for his invariants; and
- expresses them in terms of Seiberg–Witten invariants.

These lead to the determination of the wall-crossing terms and a resolution of Witten's conjecture $D = SW$ on a projective surface both by **Göttsche–Nakajima–Yoshioka** both with analysis (blowup formulae) on the Nekrasov partition function.

One may think about other insertions. **Fantechi–Göttsche** and **Ciocan-Fontanine–Kapranov**, for instance, define the *virtual Euler characteristic* of $\mathcal{M}$ by

$$e^{\mathrm{vir}}(\mathcal{M}) := \int_{[\mathcal{M}]^{\mathrm{vir}}} c_{vd}(T^{\mathrm{vir}}_{\mathcal{M}}),$$

where $T^{\mathrm{vir}}_{\mathcal{M}}$ is the virtual tagent sheaf coming form the perfect obstruction theory on $\mathcal{M}$.

More generally, they define the *virtual* χ_y*-genus* of $\mathcal{M}$ by

$$\chi^{\mathrm{vir}}_{-y}(\mathcal{M}) := \int_{[\mathcal{M}]^{\mathrm{vir}}} (1 - y\mathrm{ch}(\mathrm{T}^{\mathrm{vir}})^{-1}) \cdot \mathrm{td}(\mathrm{T}^{\mathrm{vir}})) \in \mathbb{Q}[y].$$

These virtual Euler characteristic and virtual χ_y-genus of the moduli space of semistable sheaves can be thought of as the *instanton part* of the Vafa-Witten invariant or a refinement of it.

Göttsche–Kool forms conjectures that the generating series of them and also other virtual enumerative invariants such as *virtual Segre* and *Verlinde numbers* of the moduli spaces could be written in terms of modular forms and Seiberg–Witten invariants.

They also raised conjectures on blowup formulae for them. The one for the virtual Euler characteristics of the moduli spaces resembles **Li–Qin**'s one for the virtual Hodge polynomials, whose origin is the work by Vafa–Witten, where the interesting appearance of modular forms is explained by means of Conformal Field theory, or Vertex Algebras. (cf. **Nakajima**'s pioneering works for these subjects)

A blowup formula for virtual enumerative invariants

We establish a Nakajima–Yoshioka style blowup formula for virtual enumerative invariants on a projective surface, which include the virtual Euler characteristics, virtual χ_y-genera, virtual Segre and Verlinde numbers of the moduli spaces of semistable sheaves, and the Donaldson–Mochizuki invariants, by constructing perfect obstruction theories on the moduli spaces of m-stable sheaves which interpolate the moduli space of semistable sheaves on a complex projective surface and that on its blowup at a point, via enhanced master spaces.

Nakajima–Yoshioka's *m*-stable sheaves. Let $p : \hat{X} \to X$ be the blowup at a point of a projective surface X. Fix a cohomology class $c := r + c_1 + \mathrm{ch}_2$ of X and consider $\widehat{c} = p^* c + ke$ of $\widehat{X}$, where $e := \mathrm{ch}(\mathcal{O}_C(-1))$ with C the exceptional divisor.

We would like to compare the (virtual) fundamental classes coming from the moduli spaces $\mathcal{M}_X(c)$ and $\mathcal{M}_{\widehat{X}}(\widehat{c})$ of semistable sheaves on X and $\hat{X}$ respectively.

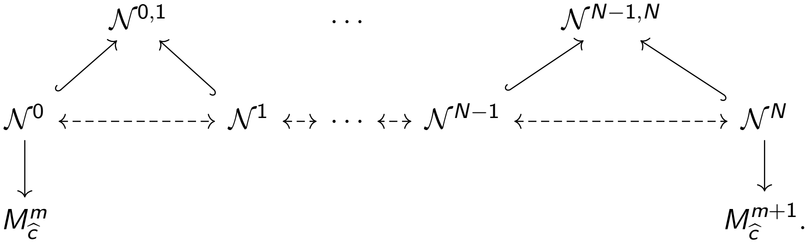
To do so, we consider the moduli spaces $M^m_{\widehat{c}}$ of m-stable sheaves on $\widehat{X}$ introduced by Nakajima–Yoshioka on the blowup, which interpolate the moduli spaces of semistable sheaves on the blowup and on the original surface in the following manner:

$$\begin{array}{ccccccccc} M^0_{\widehat{c}} & \dashleftarrow\!\dashrightarrow & M^1_{\widehat{c}} & \dashleftarrow\!\dashrightarrow & \cdots & \dashleftarrow\!\dashrightarrow & M^{m-1}_{\widehat{c}} & \dashleftarrow\!\dashrightarrow & M^m_{\widehat{c}} \\ \downarrow{\scriptstyle p_*} & & & & & & & & \downarrow \\ \mathcal{M}_X(c) & & & & & & & & \mathcal{M}_{\widehat{X}}(\widehat{c}), \end{array}$$

where the right vertical arrow becomes an isomorphism if m is sufficiently large, while so is the left vertical arrow if $\widehat{c} = p^*c$.

Wall-crossing. In order to compare things on $M_{\widehat{c}}^m$ with those on $M_{\widehat{c}}^{m+1}$, one may embed them in an intermediate space, the moduli space of $(m, m+1)$-semistable sheaves.

But, the moduli space of them has highly positive-dimensional automorphism groups, so we use Mochizuki's technique, the *enhanced moduli stack* $\mathcal{N}^{\ell}$, which is a Deligne–Mumford stack, together with the intermediate Artin stack $\mathcal{N}^{\ell,\ell+1}$, which has only one-dimensional stabiliser group, satisfying the following diagram:



We then construct the Kiem–Li style master space $\mathcal{Z}$ equipped with a $\mathbb{C}^*$-action, and its set-theoretical $\mathbb{C}^*$-fixed loci are

$$|\mathcal{Z}|^{\mathbb{C}^*} = |\mathcal{N}^{\ell}| + |\mathcal{N}^{\ell+1}| + \text{the rest.}$$

The stack-theoretical version of this with virtual integrations over the fibres of $\mathcal{N}^0$ and $\mathcal{N}^N$ leads to a wall-crossing formula we desire.

Blowup formula for Donaldson–Mochizuki invariants. One application of our wall-crossing formula is the Friedman–Morgan style blowup formula for the virtual invariants:

$$D_{\widehat{X},p^*c_1}(\sigma_1 \dots \sigma_n [C]^4) = -2D_{X,c_1}(\sigma_1, \dots, \sigma_n).$$

This leads to a direct proof of the equivalence between Mochizuki's virtual Donaldson invariants and the classical ones in our setting.

Nakajima–Yoshioka style structure theorem. Let $\Phi(\mathcal{E})$ be a Chow cohomology class depending on the universal sheaf on $X \times \mathcal{M}$ and cohomology classes of X such as the insertions for virtual Euler characteristics, virtual χ_y-genera, virtual Segre and Verlinde numbers of the moduli spaces of semistable sheaves on X, or the Donaldson–Mochizuki invariants. We then obtain:

Theorem (Kuhn–T) Fix a Chern class $\widehat{c} = p^*c - ke$ for some $k \geq 0$ on $\widehat{X}$, where $c = r + c_1 + c_2$ is that on X. Then there exist universal power series $\Omega_n \in \mathbb{Q}[[\nu_2, \dots, \nu_r]]$ depending only on r, Φ and k, which satisfy:

$$\int_{[\mathcal{M}_{\widehat{X}}(\widehat{c})]^{\mathrm{vir}}} \Phi(\mathcal{E}) = \sum_{n=0}^{\infty} \int_{[\mathcal{M}_X(c+n[pt])]^{\mathrm{vir}}} \Phi(\mathcal{E})\Omega_n(\mathcal{E}).$$

From the construction, these Ω_n can be determined by calculating them on the moduli spaces of framed sheaves on $\mathbb{P}^2$.

This may resolve conjectures by **Göttsche–Kool** and **Göttsche** on the structures of virtual Segre and Verlinde numbers of projective surfaces, and so on.

For example, we obtain the following blowup formula for the generating series of the virtual χ_y-genera of the moduli spaces:

Theorem (Kuhn–Leigh–T)

$$\sum_{\mathrm{ch}_2} \chi^{\mathrm{vir}}_{-y}(\mathcal{M}_X(c)) q^{\mathrm{vd}\,\mathcal{M}_X(c)} = Y_k(q,y) \sum_{\mathrm{ch}_2} \chi^{\mathrm{vir}}_{-y}(\mathcal{M}_{\hat{X}}(\hat{c})) q^{\mathrm{vd}\,\mathcal{M}_{\hat{X}}(\hat{c})},$$

with $$Y_k(q,y) := \frac{(q^{2r}y^r)^{r/24}}{\eta(q^{2r}y^r)^r} \left(\sum_{v \in \mathbb{Z}^{r-1} + \frac{k}{r} I} (q^{2r}y^r)^{v^t A v}\, y^{v^t A I} \right),$$

where $A = (a_{ij})$ is the $(r-1) \times (r-1)$-matrix with entries $a_{ij} = 1$ for $i \leq j$ and $a_{ij} = 0$ otherwise, and I is the column vector of length $r-1$ with all entries equal to one, η is the Dedekind eta function.

This coincides with Göttsche's conjecture on the generating series of topological χ_y-genera of the moduli spaces (hence it confirms the conjecture for the unobstructed case).